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Space-Time Groups for the Lattice[1]

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In the assumption of a lattice theory in which the continuous limit is not taken, the metric of the discrete space-time should be invariant under integral transformations. Based on local isomorphisms between real forms, a method is proposed in order to find the rational and integral elements of the pseudoorthogonal groups. Besides, the rational and integral trigonometric and hyperbolic functions are constructed on the lattice.

1. ISOMORPHISM BETWEEN REAL FORMS

According to Cartan theory, there are some real forms of simple Lie groups of low dimensionality which are locally isomorphic (Helgason, 1978). We describe them by the bijection of R^n onto a set of matrices A.

(i) $SL(2,\mathbb{R}) \approx SO(2,1)$. Define a set of 2×2 real matrices A, by the conditions $A^T = A$, where A^T means transposed. The bijection of an element (x_0, x_1, x_2) of $\mathbb{R}^3$ onto a matrix A is the following:

$$A = \begin{pmatrix} x_0 + x_2 & x_1 \\ x_1 & x_0 - x_2 \end{pmatrix}$$

The transformations $A' = SAS^T$, with $S \in SL(2,\mathbb{R})$, map A into itself. Since

$$\det A = x_0^2 - x_1^2 - x_2^2 = \det A'$$

this transformation induces the desired isomorphism.

(ii) $SL(2,\mathbb{C}) \approx SO(3,1)$. Define A, a 2×2 complex matrix, by the condition $A^+ = A$, where A^+ means the Hermitian conjugate matrix. The bijection of (x_0, x_1, x_2, x_3) in R^4 onto A is given by

$$A = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}$$

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The transformation $A'=SAS^{+}$ with $S\in SL(2,\mathbb{C})$ maps A into itself, as it is well known (Gel'fand et al., 1963). Since

$$\det A = x_0^2-x_1^2-x_2^2-x_3^2=\det A'$$

this transformation induces the mentioned isomorphism.

(iii) $Sp(4,\mathbb{R})\approx SO(3,2)$. The matrix A is a four-dimensional real matrix, satisfying $A^TJ=JA$ and $\operatorname{Tr} A=0$, where

$$J\equiv\left(\begin{array}{c:c} 0 & \mathbb{1} \\ \hdashline -\mathbb{1} & 0 \end{array}\right),$$

$\mathbb{1}$ is the unit matrix of dimension 2.

The bijection of an element $(x_1, x_2, x_3, x_4, x_5)$ of $\mathbb{R}^5$ onto A is given by

$$A=\begin{pmatrix} x_1 & x_2+x_3 & 0 & x_4+x_5 \\ x_2-x_3 & -x_1 & -x_4-x_5 & 0 \\ 0 & x_4-x_5 & x_1 & x_2-x_3 \\ -x_4+x_5 & & x_2+x_3 & -x_1 \end{pmatrix}$$

The transformation $A'=SAS^{-1}$ with $S\in Sp(4,\mathbb{R})$ maps A into itself, namely, $A'^TJ=JA'$, $\operatorname{Tr} A'=0$. Since

$$\det A=(x_1^2+x_2^2-x_3^2-x_4^2+x_5^2)^2=\det A',$$

this transformation induces the desired isomorphism.

(iv) $Sp(1,1)\approx SO(4,1)$. A is defined by the four-dimensional complex matrix satisfying $A^TJ=JA$, $A^+K=KA$, $\operatorname{Tr} A=0$, with $K\equiv \operatorname{diag}(1,-1,1,-1)$.

The bijection of an element $(x_1, x_2, x_3, x_4, x_5)$ of $\mathbb{R}^5$ onto A is

$$A=\begin{pmatrix} x_1 & x_2+ix_3 & 0 & x_4+ix_5 \\ -x_2+ix_3 & -x_1 & -x_4-ix_5 & 0 \\ 0 & x_4-ix_5 & x_1 & -x_2+ix_3 \\ -x_4+ix_5 & 0 & x_2+ix_3 & -x_1 \end{pmatrix}$$

Given an element S of the group $Sp(1,1)$, that is to say, $S^TJS=J$, $S^+KS=K$, the transformation $A'=SAS^{-1}$ maps A into itself. Since

$$\det A=(x_1^2-x_2^2-x_3^2-x_4^2-x_5^2)^2=\det A'$$

this transformation induces the desired isomorphism.

(v) $SU(2,2)\approx SO(4,2)$. A is defined by the four-dimensional complex matrix, satisfying $A^T=-A$, $A^*I=I\bar{A}$, with $\bar{A}$, the complex conjugate matrix of A, A^* the dual matrix of A, namely, $(A^*)_{ab}=\frac{1}{2}\epsilon_{abcd}A^{cd}$, and

$$I\equiv\left(\begin{array}{c:c} \mathbb{1} & 0 \\ \hdashline 0 & -\mathbb{1} \end{array}\right)$$

The bijection of an element $(x_1, x_2, x_3, x_4, x_5, x_6)$ of $\mathbb{R}^6$ onto A is (Beckers et al., 1978)

$$A=\begin{pmatrix} 0 & x_1+ix_2 & x_3+ix_4 & x_5+ix_6 \\ -x_1-ix_2 & 0 & x_5-ix_6 & -x_3+ix_4 \\ -x_3-ix_4 & -x_5+ix_6 & 0 & -x_1+ix_2 \\ -x_5-ix_6 & x_3-ix_4 & x_1-ix_2 & 0 \end{pmatrix}$$

The transformation $A'=SAS^T$, with S satisfying $S^+IS=I$, maps A into itself. Since

$$\det A=(-x_1^2-x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)=\det A'$$

this transformation belongs also to $SO(4,2)$.

(vi) $SL(2, Q)\approx SO(5,1)$. Let A be a two-dimensional quaternion matrix defined by $A^+=A$. The bijection of an element $(x_0, x_1, x_2, x_3, x_4, x_5)$ of $\mathbb{R}^6$ onto A is the following:

$$A=\begin{pmatrix} x_0+x_1 & x_2+x_3i+x_4j+x_5k \\ x_2-x_3i-x_4j-x_5k & x_0-x_1 \end{pmatrix}$$

with (i, j, k) a basis for the quaternions. The transformation $A'=SAS^+$, with $S\in SL(2, Q)$ maps A into itself. Since

$$\det A=(x_0-x_1^2-x_2^2-x_3^2-x_4^2-x_5^2)^2=\det A'$$

this transformation induces the desired isomorphism (Barut et al., 1965)

(vii) $SL(4,\mathbb{R})\approx SO(3,3)$. A is defined by the four-dimensional real matrix, satisfying $A^T=-A$. The bijection of an element $(x_1, x_2, x_3, x_4, x_5, x_6)$ of $\mathbb{R}^6$ onto A is the following:

$$A=\begin{pmatrix} 0 & -x_1+x_4 & x_2+x_5 & x_3+x_6 \\ -x_1-x_4 & 0 & x_3-x_6 & -x_2+x_5 \\ -x_2-x_5 & -x_3+x_6 & 0 & x_1-x_4 \\ -x_3-x_6 & x_2-x_5 & -x_1+x_4 & 0 \end{pmatrix}$$

The transformation $S'=SAS^T$, with $S\in SL(4,\mathbb{R})$ maps A into itself. Since

$$\det A=(x_1^2+x_2^2+x_3^2-x_4^2-x_5^2-x_6^2)^2=\det A'$$

this transformation induces the desired isomorphism.

2. RATIONAL AND INTEGRAL REAL FORMS

Using the Cayley parametrization of simple Lie groups we have described a method to find all elements of a classical group with rational matrix elements in the fundamental representation (Lorente, 1974). In this method a matrix S of the unitary, orthogonal or symplectic group is decomposed in the following way:

$$S=\frac{1-H}{1+H}=\frac{(1-H)^2}{1-H^2}$$

where H satisfies some particular conditions. S will be a rational form if the matrix elements of H take only integer values. A rational form transforms vectors with rational components into other vectors with rational components.

If we impose on H the condition to be a nilpotent matrix (Patera et al., 1980) $H^2=0$, the last equation becomes

$$S=1-2H$$

with the property $S^k=1-2kH$, k integer. In this case, if the matrix elements of H take only integer values then S will be an integral real form, which transforms vectors with (Gaussian) integer components into vectors with (Gaussian) integer components. In the case of an N-dimensional hypercubic lattice of a Euclidean space $\mathbb{R}^N$ these transformations map the lattice into itself. In the case of unimodular groups we can also decompose $S=1+H$, with $H^2=0$, since in this case $\det S=1$. We apply this method to the real forms of the last section in order to find more easily the rational and integral forms of the corresponding orthogonal groups.

(i) $SL(2,\mathbb{R})$: $\det S=1$; integral forms $S=1+H$, $H^2=0$:

$$H=\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix},\quad \begin{pmatrix}0 & 0\\ b & 0\end{pmatrix},\quad \begin{pmatrix}c & c\\ -c & -c\end{pmatrix},\qquad a, b, c \text{ integers}$$

(ii) $SL(2,\mathbb{C})$ $\det S=1$; integral forms: $S=1+H$, $H^2=0$:

$$H=\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix},\quad \begin{pmatrix}0 & 0\\ b & 0\end{pmatrix},\quad \begin{pmatrix}c & c\\ -c & -c\end{pmatrix},$$

with a, b, c Gaussian integers.

(iii) $Sp(4,\mathbb{R})$, $H^TJ+JH=0$

$$H=\begin{pmatrix}a_1 & a_2 & b_1 & b_2\\ a_3 & a_4 & b_2 & b_3\\ c_1 & c_2 & -a_1 & -a_3\\ c_2 & c_3 & -a_2 & -a_4\end{pmatrix}$$

Rational forms: H with integer entries; integral forms: $H^2=0$, such as $a_1=a_2=-a_3=-a_4=1$; $b_1=b_2=b_3=b_4=c_1=c_2=c_3=c_4=0$.

(iv) $Sp(1,1)$. $H^TJ+JH=0$, $H^+K+KH=0$. Rational forms: H with Gaussian integer numbers; integral forms: $H^2=0$, as in (iii).

(v) $SU(2,2)$ $H^{+}I+IH=0$; with the unitary transformation $I \to UIU^{-1}$, such that

$$I=\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

we obtain the following integral transformation

$$H=\begin{pmatrix} h_1 & \\ & -h_1^{+} \end{pmatrix}, \quad H=\begin{pmatrix} 0 & h_2 \\ 0 & 0 \end{pmatrix}, \quad H=\begin{pmatrix} 0 & 0 \\ h_2 & 0 \end{pmatrix}, \quad h_1^2=0, \quad h_2^{+}=h_2$$

corresponding to pure Lorentz, translation, and pure conformal transformations (Mack, 1977).

(vi) $SL(2,Q)$: $\det S=1$. Integral forms: $S=1+H$, with $H^2=0$; for instance,

$$H=\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, \quad \begin{pmatrix} c & c \\ -c & -c \end{pmatrix}, \quad a, b, c \in Q, \text{ integers quaternions}$$

(vii) $SL(4,\mathbb{R})$, $\det S=1$. Integral forms: $S=1+H$, $H^2=0$, with integer matrix elements for H.

(viii) For a pseudoorthogonal group $SO(n,n)$ the Cayley parametrization gives $H^{T}I+IH=0$ with

$$H=\begin{pmatrix} h_1 & \vdots & h_2 \\ \cdots & \vdots & \cdots \\ h_3 & \vdots & -h_1^{T} \end{pmatrix}, \qquad h_2^{T}=-h_2, \qquad h_3^{T}=-h_3$$

Rational forms: H with integer elements; integral forms: $H^2=0$; for instance

$$h_1^2=0, \quad h_2=h_3=0; \qquad h_1=h_3=0; \qquad h_1=h_2=0$$

Two observations can be made: First, by this method we do not get all the integral forms. To our knowledge, this problem has been solved for Lorentz transformations only (Schild, 1948). Secondly, from the rational real forms we can recover the continuous transformations, if we can take the n power of some element of the group

$$S=\frac{[1-(1/m)H]^n}{[1+(1/m)H]^n}$$

which is still an element of the same group. (Here all the parameters are divided by m for convenience). Now, if we make $m=n$, and take the limit $n\to\infty$, we get

$$S=e^{-2H}$$

3. RATIONAL TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

We have defined (Lorente, 1974) the rational functions

$$\cos k\alpha = \frac{1}{2}\left(\frac{z^{2k}}{|z|^{2k}} + \frac{\bar{z}^{2k}}{|z|^{2k}}\right), \qquad \sin k\alpha = \frac{1}{2i}\left(\frac{z^{2k}}{|z|^{2k}} - \frac{\bar{z}^{2k}}{|z|^{2k}}\right)$$

with $z = m + ni$, m, n, k integers, and α represent twice the area of the circular sector limited by the vectors P_k and P_{k+1}, ($P_k \equiv z^{2k}/|z|^{2k}$). In order to rationalize the argument also, we take α to be the length of the chord between P_k and P_{k+1}, namely,

$$l_0^2 \equiv \alpha^2 = \left|\frac{z^2}{|z|^2} - 1\right|^2 = \frac{4n^2}{m^2+n^2} = 2(1 - \cos\alpha)$$

Then the rational trigonometric functions satisfy the difference equations:

$$\frac{\Delta^2 \cos k\alpha}{\Delta\alpha^2} + \cos(k+1)\alpha = 0, \qquad \frac{\Delta^2 \sin k\alpha}{\Delta\alpha^2} + \sin(k+1)\alpha = 0$$

when we have written $\Delta\alpha^2$ instead of α^2 for analogy with the differential equations. We have also defined the rational hyperbolic functions

$$\cosh k\beta = \frac{1}{2}\left(\frac{u^{2k}}{|u|^{2k}} + \frac{\bar{u}^{2k}}{|u|^{2k}}\right), \qquad \sinh k\beta = \frac{1}{2}\left(\frac{u^{2k}}{|u|^{2k}} - \frac{\bar{u}^{2k}}{|u|^{2k}}\right)e_1$$

with $u = m + ne_1$, m, n, k integers, $e_1^2 = 1$. In order to rationalize the argument, we take β to be the length of the chord with negative sign between two vectors P_k and P_{k+1} ($P_k \equiv u^{2k}/|u|^{2k}$):

$$l_0^2 \equiv -\beta^2 = \left|\frac{u^2}{|u|^2} - 1\right|^2 = -\frac{4n^2}{m^2-n^2} = 2(1 - \cosh\beta)$$

Then, the difference equations take the form

$$\frac{\Delta^2 \cosh k\beta}{\Delta\beta^2} - \cosh(k+1)\beta = 0, \qquad \frac{\Delta^2 \sinh k\beta}{\Delta\beta^2} - \sinh(k+1)\beta = 0$$

with $\Delta\beta^2$ instead of β^2 for analogy with the differential equations. If we define

$$\exp k\beta \equiv \cosh k\beta + \sinh k\beta = (\cosh\beta + \sinh\beta)^k$$

we get also $\Delta^2 \exp k\beta/\Delta\beta^2 - \exp(k+1)\beta = 0$.

We have defined also the generalized trigonometric functions (Lorente, 1974)

$$\cos k\alpha = \frac{1}{2}\left(\frac{w^{2k}}{|w|^{2k}} + \frac{\bar{w}^{2k}}{|w|^{2k}}\right), \qquad k \text{ integer}$$

$$\sin_i k\alpha = \frac{1}{2}\left(\frac{w^{2k}}{|w|^{2k}} u_i - u_i \frac{\bar{w}^{2k}}{|w|^{2k}}\right), \qquad i = 1, 2, 3$$

where

$$w = m + nu_1 + pu_2 + qu_3, \qquad m, n, p, q \text{ integers}$$

$$\bar{w} = m - nu_1 - pu_2 - qu_3, \qquad |w|^2 = w\bar{w}$$

are quaternion numbers. We rationalize the argument of these functions, as before,

$$\alpha^2 = l_0^2 = \left|\frac{w^2}{|w|^2} - 1\right|^2 = 4\frac{m^2 + p^2 + q^2}{m^2 + n^2 + p^2 + q^2} = 2(1 - \cos\alpha)$$

and the difference equations become

$$\frac{\Delta^2 \cos k\alpha}{\Delta\alpha^2} + \cos(k+1)\alpha = 0, \qquad \frac{\Delta^2 \sin_i k\alpha}{\Delta\alpha^2} + \sin_i(k+1)\alpha = 0$$

$i = 1, 2, 3$ and $\Delta\alpha^2$ instead of α^2.

For the generalized hyperbolic function we define

$$\cosh k\beta = \frac{1}{2}\left(\frac{u^{2k}}{|u|^{2k}} + \frac{\bar{u}^{2k}}{|u|^{2k}}\right), \qquad k \text{ integer}$$

$$\sinh_i k\beta = \frac{1}{2}\left(\frac{u^{2k}}{|u|^{2k}} e_i - e_i \frac{\bar{u}^{2k}}{|u|^{2k}}\right), \qquad i = 1, 2, 3$$

where

$$u = m + re_1 + se_2 + te_3, \qquad m, r, s, t \text{ integers}$$

$$\bar{u} = m - re_1 - se_2 - te_3, \qquad |u|^2 = u\bar{u}$$

and e_1, e_2, e_3 satisfy $e_i e_j + e_j e_i = \delta_{ij}$.

We take for the argument β the length of the chord with negative sign between the vectors P_k and P_{k+1} ($P_k \equiv u^{2k}/|u|^{2k}$), namely,

$$-\beta^2 \equiv l_0^2 = \left|\frac{u^2}{|u|^2} - 1\right|^2 = -4\frac{r^2 + s^2 + t^2}{m^2 - r^2 - s^2 - t^2} = 2(1 - \cosh\beta)$$

Hence the difference equations become

$$\frac{\Delta^2 \cosh k\beta}{\Delta\beta^2} - \cosh(k+1)\beta = 0, \qquad \frac{\Delta^2 \sinh k\beta}{\Delta\beta^2} - \sinh(k+1)\beta = 0$$

($i = 1, 2, 3$) with $\Delta\beta^2$ instead of β^2.

As in the case of rational forms, we can recover the continuous limit of a rational function. Take the m power of the exponential functions, for instance:

$$\exp m\beta = \left(\frac{m+n}{m-n}\right)^m = \left(\frac{1+n/m}{1-n/m}\right)^m$$

which becomes e^{-2n}, when $m \to \infty$.

4. INTEGRAL TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

With the help of the integral real forms of the orthogonal group, we can construct the integral trigonometric functions. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

be an $N \times N$ orthogonal matrix such that $A^N = \mathbb{1}$. Consider the matricial function

$$A(x) = A^x, \qquad x \text{ integer}$$

with $A(0) = A(N) = \mathbb{1}$, satisfying the periodicity condition

$$A(x+N) = A(x)$$

Using the matrix elements of this function we define

$$a_{ij}(x) \equiv [A(x)]_{ij}, \qquad i, j = 1, 2, \ldots, N$$

From the properties of $A(x)$ we derive immediately the following properties:

(i) $a_{ij}(x+y) = \sum_{m=1}^{N} a_{im}(x) a_{mj}(y), \qquad x, y$ integers

(ii) $\sum_{j=1}^{N} a_{ij}^2(x) = 1, \qquad i = 1, 2, \ldots, N$

(iii) $\sum_{i=1}^{N} a_{ij}^2(x) = 1, \qquad j = 1, 2, \ldots, N$

(iv) $a_{ij}(x+N) = a_{ij}(x)$

We can construct also the difference operators

$$\Delta a_{ij}(x) = a_{ij}(x+1) - a_{ij}(x) \equiv (E-1) a_{ij}(x)$$

satisfying

$$E^k a_{ij}(x) = a_{i,j+k(\mathrm{mod}\,N)}(x) \qquad E^N a_{ij}(x) = a_{ij}(x)$$

We can also define the "Fourier" expansion of some periodic integral function, of period N,

$$f(x) = f(x+N), \qquad x \text{ integer}$$

in terms of the integral trigonometric function,

$$f(x) = \sum_{j=1}^{N} c_j a_{ij}(x), \qquad (i \text{ fixed})$$

With the help of the orthogonality condition

$$\sum_{x=1}^{N} a_{ij}(x) a_{ik}(x) = \delta_{jk}, \qquad (i \text{ fixed})$$

we obtain

$$c_k = \sum_{x=1}^{N} f(x) a_{ik}(x), \qquad (i \text{ fixed})$$

We can construct also the integral hyperbolic functions with the help of the vector

$$v = v_0 + v_1 e_1 + v_2 e_2 + v_3 e_3$$

where $v_1^2 + v_2^2 + v_3^2 - v_0^2 = 1$. Hence

$$\cosh k\beta = v^{2k} + v^{-2k}, \qquad \sinh_i k\beta = v^{2k} e_i - e_i v^{-2k}, \qquad i = 1, 2, 3$$

k integer, and the argument β defined as the length of the chord, with negative sign, between the vectors v^{2k+2} and v^{2k},

$$-\beta^2 \equiv l_0^2 = |v^2 - 1|^2 = -4v_0^2$$

Therefore $\beta = 2v_0$ is an integer number. The integral hyperbolic functions satisfy

$$\Delta^{2l} \cosh k\beta - \beta^{2l} \cosh(k+1) = 0$$

and for k, negative, $k = -1$, we obtain

$$\Delta^{2l} \cosh k\beta = \beta^{2l}$$

which can be used for the expansion of an integral function in terms of the integral hyperbolic functions.

5. SOME COMMENTS

We have presented some mathematical aspects of a physical model based on the hypothesis of a discrete space and time. The philosophical foundations for this model were presented elsewhere (Lorente 1983). From different approaches this hypothesis has become very appealing:

(i) In Tutzing seminars, the hypothesis of simple alternatives, elementary processes, monads, and similar ones, require some discrete structure for the configuration or momentum space (Castell et al., 1975, 1977, 1979, 1981, 1983).

(ii) In relational theories of space and time, the metric is a consequence of the intrinsic properties of the structure of space and time, as it was suggested by Riemann in the last century and has been proposed recently (Gruenbaum, 1977).

(iii) In current models of gauge theories on the lattice, although it is considered the discretization of the space as an auxiliary tool, it has been suggested that the continuous limit is only an approximation and that the real world is discrete (Klapunovsky, 1985).

(iv) One of the most difficult setbacks of discrete models is the invariance of the lattice under the space-time groups. We have found a large class of subgroups of the real forms that keep the lattice invariant. In the case of the Lorentz group this method was proposed by the author (Lorente, 1974).

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